Sankaku Project Memo

- Cylindrical Projection Map to Tetrahedron Globe -

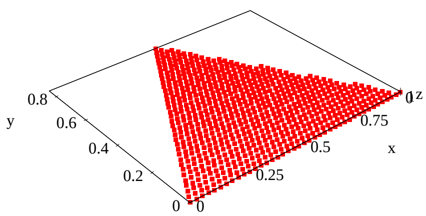
1 What's this ?

This is a memo to remind me that how a program sankaku makes tetrahedron globe from cylindrical projection map. The program is available from my homepage http://www2f.biglobe.ne.jp/~notchi/ .

2 Defining Base Triangle

Define a right triangle has a side length 1 and a bottm side lies (0,0,0) to (1,0,0). Let this triangle be p_0.

Figure 1 : Triangle p0



3 Constructing Four Triangles

3.1 Triangle p_1

Transfer the triangle to p_1. p_1 is obtained following matrix calculation.

- Rotate it $\pi + \cos^{-1}\frac{1}{3}$ radians around x axis.

$$p_1' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\pi + \cos^{-1}\frac{1}{3}) & \sin(\pi + \cos^{-1}\frac{1}{3}) \\ 0 & -\sin(\pi + \cos^{-1}\frac{1}{3}) & \cos(\pi + \cos^{-1}\frac{1}{3}) \end{pmatrix} p_0$$

- Rotate it $\frac{\pi}{6}$ radians around z axis

$$p_1'' = p_1' \begin{pmatrix} \cos\frac{2\pi}{3} & \sin\frac{2\pi}{3} & 0 \\ -\sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

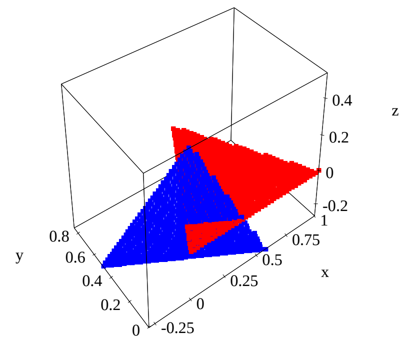
- Move it this much

$$p_1^{(3)} = p_1'' + \begin{pmatrix} -f\sin\frac{\pi}{6} \\ f\cos\frac{\pi}{6} \\ 0 \end{pmatrix} \text{ where } f = \frac{1}{2}\cos\frac{\pi}{6}$$

- Move it again this much

$$p_1 = p_1^{(3)} + \begin{pmatrix} 0 \\ 0 \\ -\frac{\sqrt{6}}{3} \end{pmatrix}$$

Figure 2 : Triangle p0 (red) and p2 (blue)

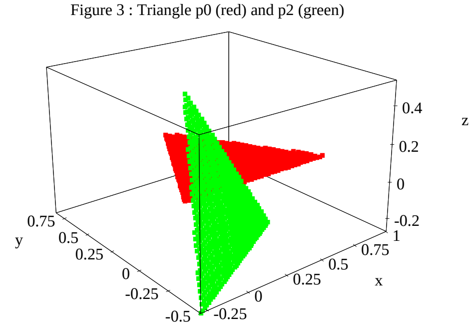


3.2 Triangle p_2

To obtain p_2, p_1 is rotated and moved by following Matrix Calculation p_2 is Rotate p_1 $\frac{2\pi}{3}$ radians around z axis. In other words,

$$p_2 = \begin{pmatrix} \cos\frac{2\pi}{3} & \sin\frac{2\pi}{3} & 0 \\ -\sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix} p_1$$

Figure 3 : Triangle p0 (red) and p2 (green)

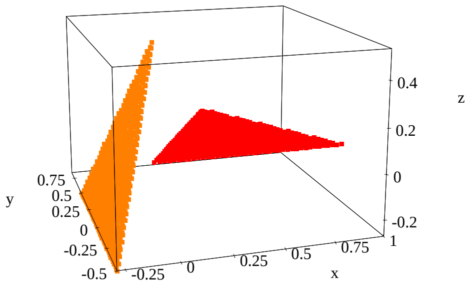


3.3 Triangle p_3

Similary to p_2, Triangle p_3 is obtained by rotating p_1 $-\frac{2\pi}{3}$ radians around z axis. In other words,

$$p_3 = \begin{pmatrix} \cos\frac{-2\pi}{3} & \sin\frac{-2\pi}{3} & 0 \\ -\sin\frac{-2\pi}{3} & \cos\frac{-2\pi}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix} p_1$$

Figure 4 : Triangle p0 and p3



3.4 Triangle p_4

Triangle p_4 is obtained by following rotation and movement of p_0.

- Rotate $\frac{-\pi}{6}$ radians around z axis

$$p_4' = \begin{pmatrix} \cos\frac{-\pi}{6} & \sin\frac{-\pi}{6} & 0 \\ -\sin\frac{-\pi}{6} & \cos\frac{-\pi}{6} & 0 \\ 0 & 0 & 1 \end{pmatrix} p_0$$

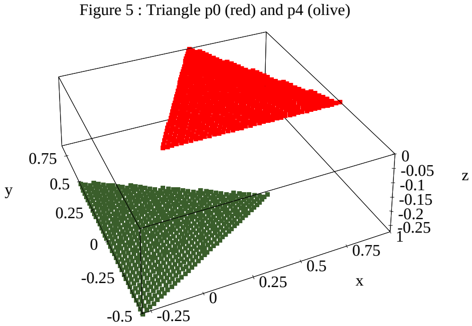
- Move it this much

$$p_4'' = p_4' + \begin{pmatrix} \frac{\sqrt{3}}{6} \\ \frac{-1}{2} \\ 0 \end{pmatrix}$$

- Move it $\frac{-\sqrt{6}}{3}$ along z axis.

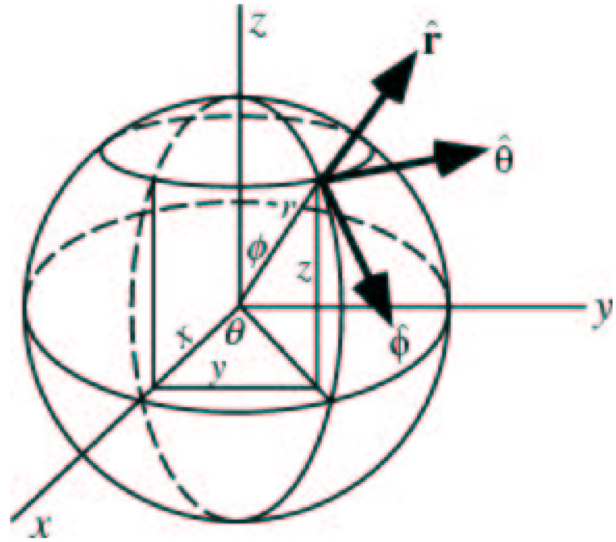
$$p_4 = p_4'' + \begin{pmatrix} 0 \\ 0 \\ \frac{-\sqrt{6}}{3} \end{pmatrix}$$

Figure 5 : Triangle p0 (red) and p4 (olive)



4 Spherical Coordinate

We've got right tetrahedron with a side length 1 and origin centered. Now we project it to sphere with radius 1 by using spherical coordinate. Refere to figure 6 for variable θ and ϕ.



- Defining unit vector $\hat{x} < 1, 0, 0 >$.
- Projection of vector $< x, y, z >$ onto $x - y$ plane is r_{xy}. So

$$\vec{r_{xy}} =< x, y, 0 >$$

- Angle θ is obtained by dot product

$$\begin{aligned} \hat{x} \cdot \vec{r_{xy}} &= |\hat{x}||\vec{r_{xy}}| \cos\theta \\ \theta &= \cos^{-1} \frac{\hat{x} \cdot \vec{r_{xy}}}{|\vec{r_{xy}}||\hat{x}|} \\ &= \cos^{-1} \frac{x}{\sqrt{x^2 + y^2}} \end{aligned}$$

  if $y < 0$, $\theta = -\theta$. Since we don't use regular procedure to convered cartesian coordinate to spherical coordinate due to $tan^{-1}\frac{x}{y}$ returns $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, it produces more branch.

- ϕ is

$$\begin{aligned} \phi &= \cos^{-1} \frac{z}{r} \\ &= \cos^{-1} \frac{z}{\sqrt{x^2 + y^2 + z^2}} \end{aligned}$$

- Finally, we get

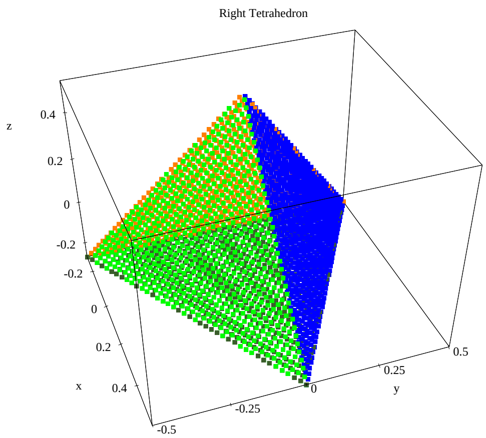
  if $y \geq 0$

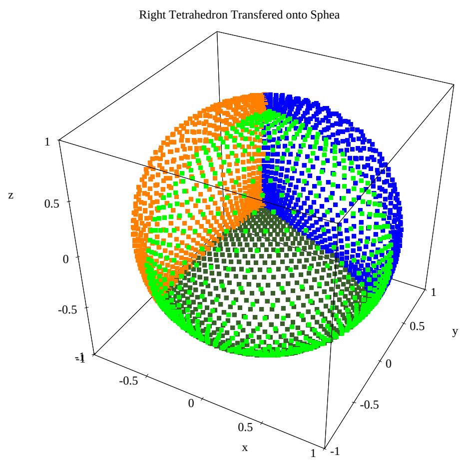
$$(\theta, \phi, r) = \left(\cos^{-1} \frac{x}{\sqrt{x^2 + y^2}}, \cos^{-1} \frac{z}{\sqrt{x^2 + y^2 + z^2}}, 1\right)$$

  if $y < 0$

$$(\theta, \phi, r) = \left(-\cos^{-1} \frac{x}{\sqrt{x^2 + y^2}}, \cos^{-1} \frac{z}{\sqrt{x^2 + y^2 + z^2}}, 1\right)$$

  why $r = 1$ is that because we are making sphere with radius 1.

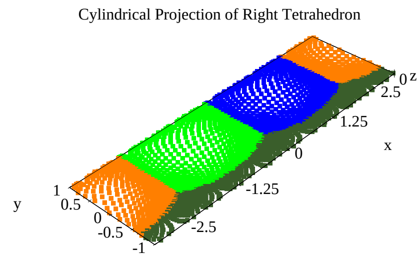




5 Cylindrical Coordinate Map

Take cylindrical projection of the sphere makes the world atlas map of the sphea. In this map, the vertical axis represents latitude and the horizontal axis represents longitude. So the conversion is here.

$$x_{map} = \theta$$
$$y_{map} = \sin \frac{\pi}{2} - \phi$$



□